Explosion speed of continuous state branching processes indexed by the Esscher transform

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Based on a work in collaboration with Loïc Chaumont

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Definition of CSBP

We consider Z a Continuous State Branching Process (CSBP) : Z is a strong Markov process wich takes value in $\mathbb{R}_+ \cup \{\infty\}$ with the following branching property :

$$\forall \lambda, t, x, y \geqslant 0,\ \mathbb{E}_{x+y}\left(e^{-\lambda Z_t}\right) = \mathbb{E}_x\left(e^{-\lambda Z_t}\right)\mathbb{E}_y\left(e^{-\lambda Z_t}\right),$$

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and u is a solution of

$$\int_{u_t(\lambda)}^{\lambda} \frac{du}{\varphi(u)} = t.$$

where φ is the Laplace exponent of a spectrally positive Lévy process.

Lamperti Representation

Let X be a spectrally positive Lévy process with Laplace exponent φ. We define $I_t := \inf\left\{s \geqslant 0, \int_0^s \frac{du}{X_u} > t\right\}$, and $\tau := \{t \geqslant 0, X_t \leqslant 0\}$. Then $Z_t := X_{I_t \wedge \tau}$ defines a branching process with branching mecanism φ.

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Z can hit ∞ in a finite if the Grey condition :

$$\int_{0+} \frac{du}{|\varphi(u)|} < \infty$$

is satisfied. We now consider φ that satisfies this condition and we denote $\zeta = \inf\{t \geqslant 0, Z_t = \infty\}$.

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- With Lamperti representation, we associate a (conservative) CSBP $Z^{(n)}$ to $X^{(n)}$.
- We study the first passage time of $Z^{(n)}$ above a level that tends to infinity.

Esscher transform

Let $(\varepsilon_n)_{n\geqslant 0}$ be a decreasing sequence of real numbers such that $\lim_n \varepsilon_n = 0$.

- For $n \geqslant 0$, we define $\varphi^{(n)}(\lambda) = \varphi(\lambda + \varepsilon_n) - \varphi(\varepsilon_n)$.

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- Then on the same probability space, we can define $(S^{(n)}, n \geqslant 0)$ a sequence of independent Lévy processes such that $S^{(n)}$ has $\varphi^{(n+1)} - \varphi^{(n)}$ for Laplace exponent, and $X^{(0)}$ with Laplace exponent $\varphi^{(0)}$.

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The sequence $(X^{(n)})$ converges uniformly over any closed intervals of $\mathbb{R}_+$, almost surely, toward a Lévy process X whose Laplace exponent is φ.

Construction of $\left(Z^{(n)}\right)$

For each $n \geqslant 0$, we define $I_t^{(n)} = \inf\left\{s \geqslant 0, \int_0^s \frac{du}{X_u^{(n)}} > t\right\}$, and $Z_t^{(n)} = X^{(n)}_{I_t^{(n)} \wedge \tau^{(n)}}$.

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Note that for all $n \geqslant 0$, $\int_{0+} \frac{du}{|\varphi^{(n)}(u)|} = \int_{0+} \frac{du}{|\varphi(u+\varepsilon_n) - \varphi(\varepsilon_n)|} = \infty$, and then $Z^{(n)}$ does not hit ∞ in a finite time.

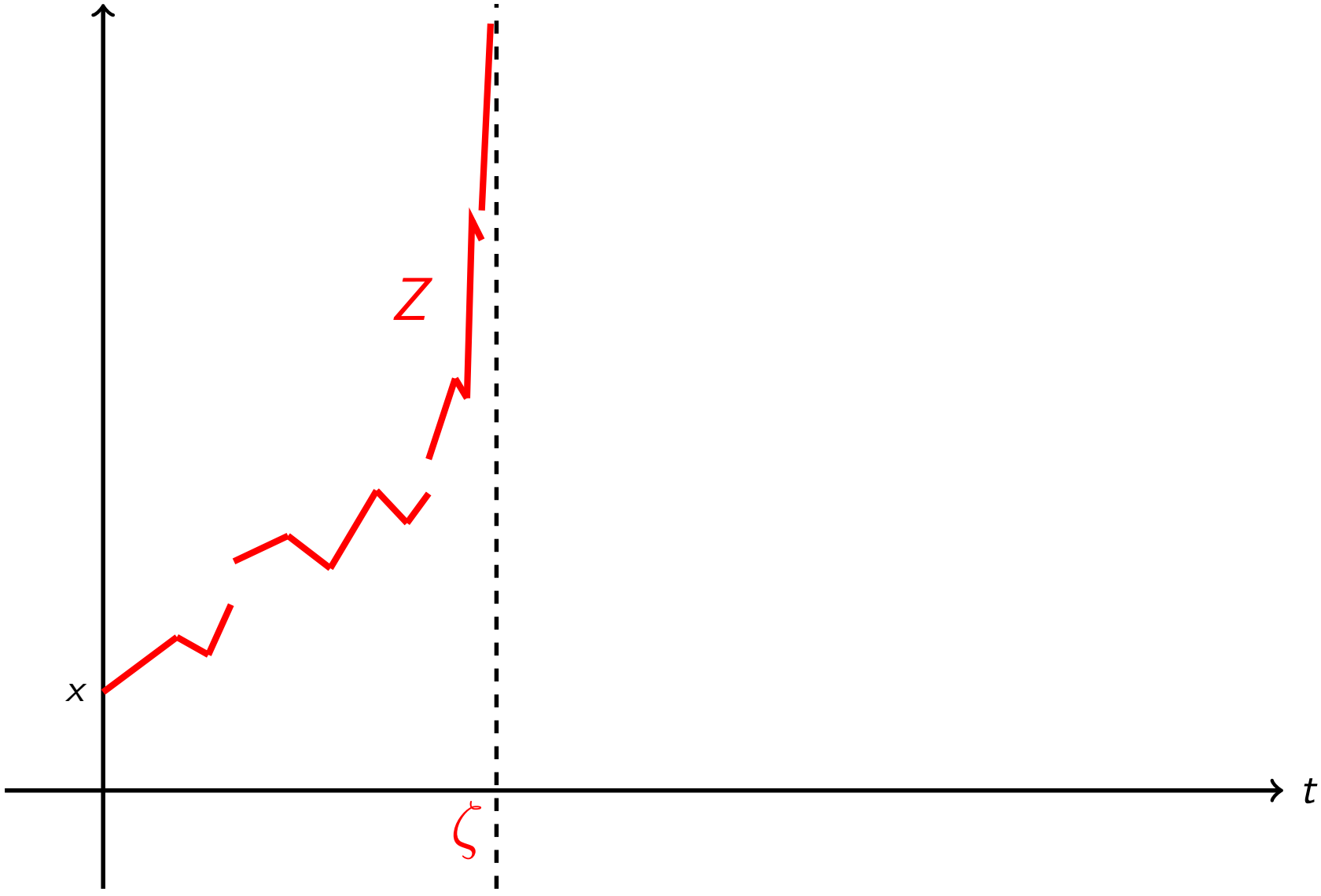
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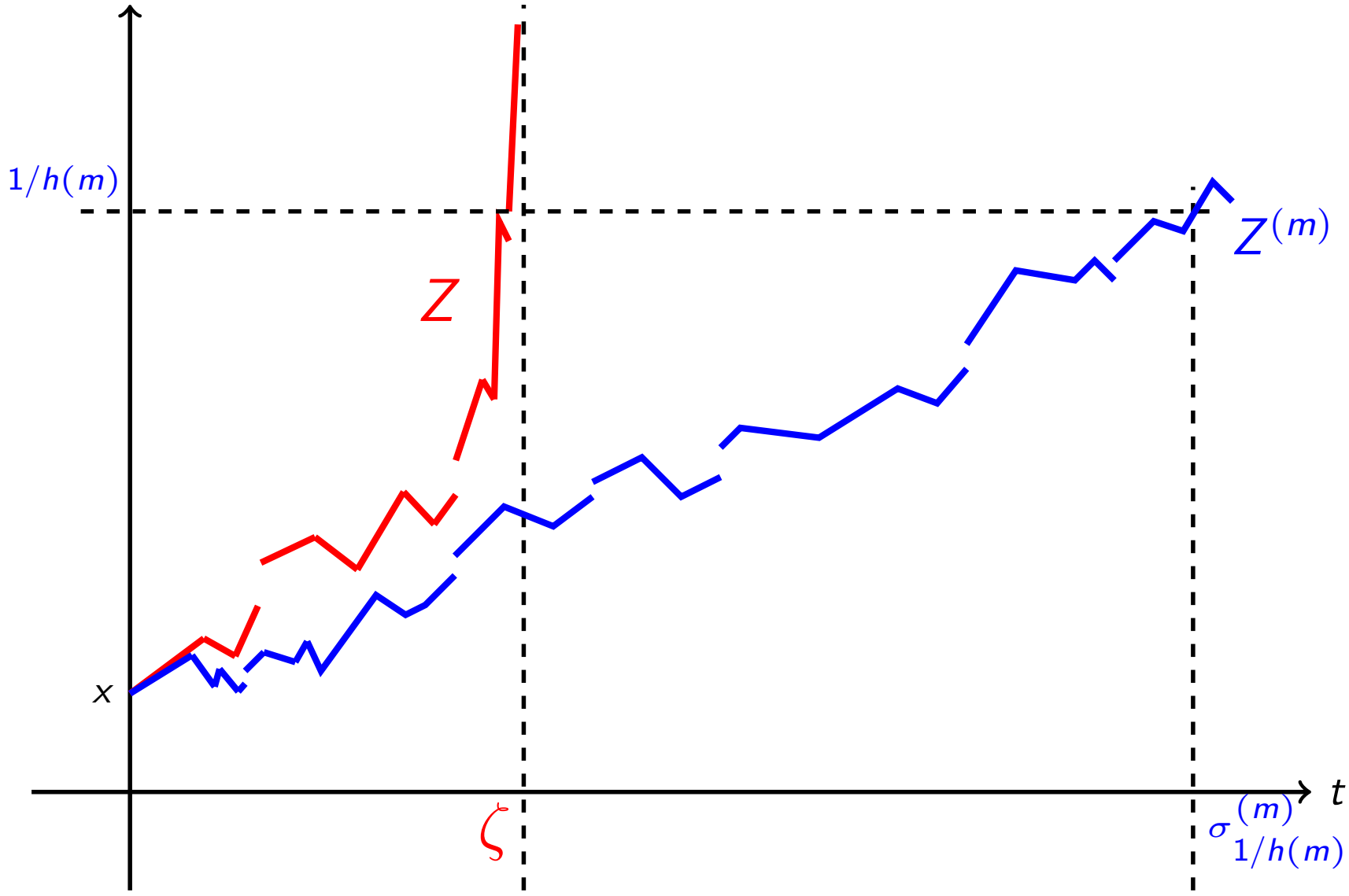
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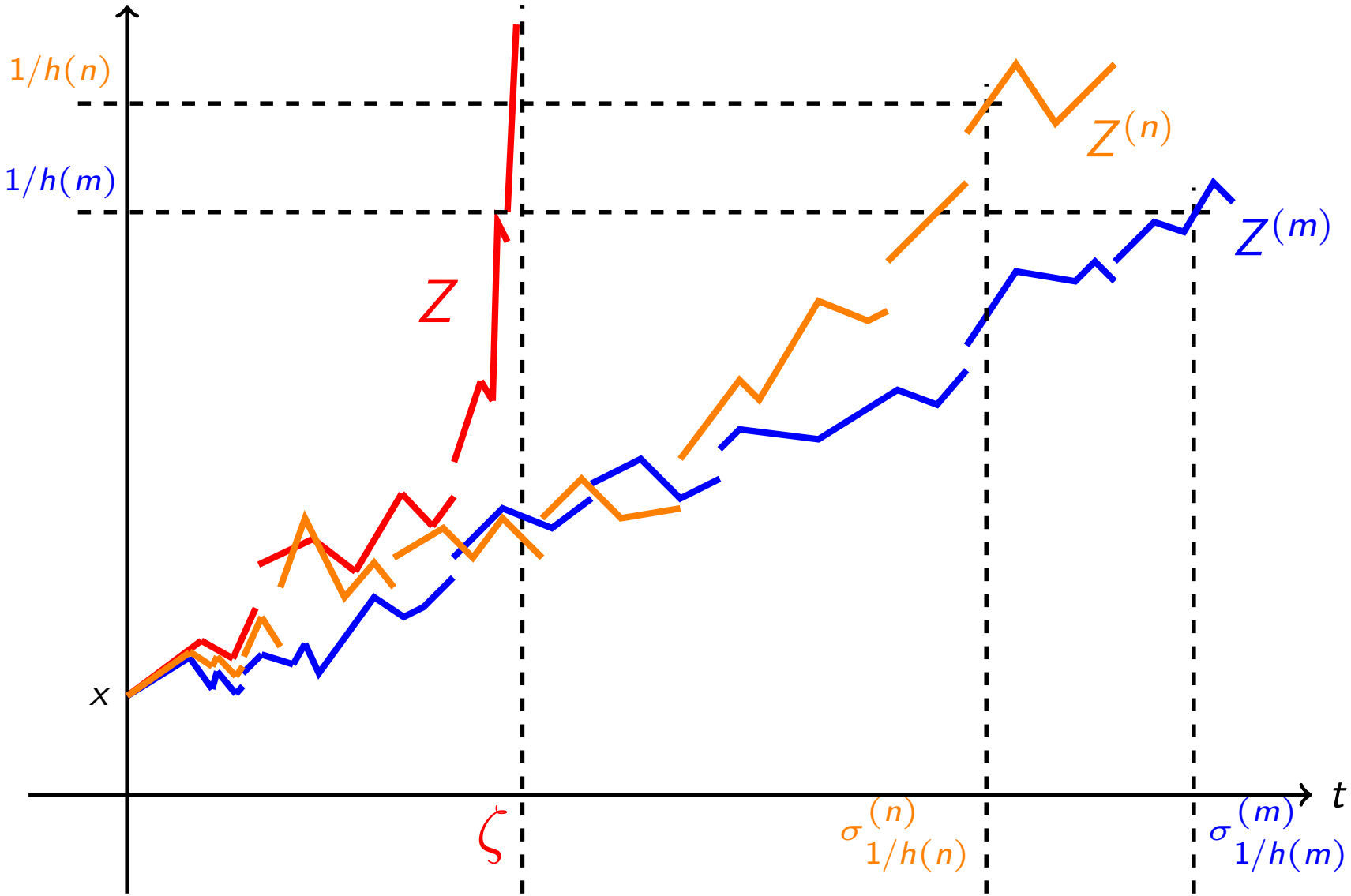
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We consider a function $h : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $(h(n))_{n \geqslant 0}$ is a decreasing sequence and $\lim_{n\to\infty} h(n) = 0$. We define $\sigma_a^{(n)} = \inf\{t \geqslant 0, Z_t^{(n)} > a\}$. We will study the sequence $(\sigma^{(n)}_{1/h(n)}, n \geqslant 0)$.


Z
x
t
ζ


$1/h(m)$
Z
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x
t
ζ
$\sigma^{(m)}_{1/h(m)}$


$1/h(n)$
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Convergence in law

Theorem

The following assertions are equivalent :

1. There is a $[0,\infty]$-valued random variable Y such that for all $k>0$ and for all $x \geqslant 0$, under $\mathbb{P}_x$, $\sigma^{(n)}_{k/h(n)} \xrightarrow[n\to\infty]{(d)} Y$.
2. For all $\theta \in (0,\rho)$, $\lim\limits_{n\to\infty} \int_{h(n)}^{\theta} \dfrac{du}{-\varphi^{(n)}(u)}$ exists.

When the above conditions are satisfied, for all $x \geqslant 0$, under $\mathbb{P}_x$,

$$Y \overset{(d)}{=} \zeta + c(h) \quad \text{where} \quad \lim_{n\to\infty} \int_{h(n)}^{\theta} \frac{du}{-\varphi^{(n)}(u)} = \int_0^{\theta} \frac{du}{-\varphi(u)} + c(h)\,,$$

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When φ is regularly varying at $0+$, these conditions are equivalent to :
$c = \lim\limits_{n\to\infty} \dfrac{\log h(n)}{\varphi'(\varepsilon_n)}$ exists, and then $c(h) = c$.

L1 and almost sure convergence

Theorem

Assume that for all $k > 0$, $\sigma^{(n)}_{k/h(n)} \xrightarrow[n\to\infty]{(d)} \zeta$. Then for all $x > 0$ and $n \geqslant 0$, the random variable $\sigma^{(n)}_{1/h(n)} \mathbb{1}_{\{\sigma^{(n)}_{1/h(n)} < \infty\}}$ is integrable under $\mathbb{P}_x$, moreover,

$$\sigma^{(n)}_{1/h(n)} \mathbb{1}_{\{\sigma^{(n)}_{1/h(n)} < \infty\}} \xrightarrow[n\to\infty]{\mathrm{L}^1(\mathbb{P}_x)} \zeta \mathbb{1}_{\{\zeta < \infty\}} .$$

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Theorem

Assume that $\sum \phi(\varepsilon_n)/\phi^{(n)}(h(n)) < \infty$. Then for all $x > 0$,

$$\sigma^{(n)}_{1/h(n)} \mathbb{1}_{\{\sigma^{(n)}_{1/h(n)} < \infty\}} \xrightarrow[n\to\infty]{\mathbb{P}_x - a.s.} \zeta \mathbb{1}_{\{\zeta < \infty\}} .$$

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